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Capacities, surface area, and radial sums ☆

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Abstract

A dual capacitary Brunn–Minkowski inequality is established for the $(n-1)$-capacity of radial sums of star bodies in $\mathbb{R}^n$. This inequality is a counterpart to the capacitary Brunn–Minkowski inequality for the p-capacity of Minkowski sums of convex bodies in $\mathbb{R}^n$, $1 \leqslant p < n$, proved by Borell, Colesanti, and Salani. When $n \geqslant 3$, the dual capacitary Brunn–Minkowski inequality follows from an inequality of Bandle and Marcus, but here a new proof is given that provides an equality condition. Note that when $n = 3$, the $(n-1)$-capacity is the classical electrostatic capacity. A proof is also given of both the inequality and a (different) equality condition when $n = 2$. The latter case requires completely different techniques and an understanding of the behavior of surface area (perimeter) under the operation of radial sum. These results can be viewed as showing that in a sense $(n-1)$-capacity has the same status as volume in that it plays the role of its own dual set function in the Brunn–Minkowski and dual Brunn–Minkowski theories.

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1. Introduction

This paper focuses on two fundamental ingredients of mathematics: the Brunn–Minkowski inequality, one of the most powerful inequalities in analysis and geometry, and the electrostatic capacity (or more generally, p-capacity) of a set in $\mathbb{R}^n$.

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The *Brunn–Minkowski inequality* for convex bodies K and L in $\mathbb{R}^n$ states that

$$\mathcal{H}^n(K+L)^{1/n} \geqslant \mathcal{H}^n(K)^{1/n} + \mathcal{H}^n(L)^{1/n}, \tag{1}$$

where $K+L$ is the Minkowski or vector sum of K and L, $\mathcal{H}^n$ denotes n-dimensional Hausdorff (or, equivalently, Lebesgue) measure, and equality holds if and only if K is homothetic to L. (See Section 2 for unexplained notation and terminology.)

It is known that (1) still holds when the sets concerned are Lebesgue measurable, and indeed the Brunn–Minkowski inequality reaches far beyond geometry. No less than three recent surveys cover its extensive generalizations, variations, connections, and applications in probability and statistics, information theory, Banach space theory, algebraic geometry, geometric tomography, interacting gases, and crystallography; see [4,14,33].

The Brunn–Minkowski inequality (1) is a cornerstone of the vast Brunn–Minkowski theory, expounded in [36]. This harbors the tools, such as Minkowski sum, for metrical problems on convex bodies and their projections onto subspaces. Around 1975, Lutwak [30] observed that when the Minkowski sum of two sets is replaced by an operation he called radial sum, in which only sums of parallel vectors are taken into account, a theory arises that is ideal for treating metrical problems about sets star-shaped at the origin, and their intersections with subspaces. This newer theory, now called the dual Brunn–Minkowski theory, has attracted much attention and counts among its successes the solution of the 1956 Busemann–Petty problem on volumes of central sections of origin-symmetric convex bodies; see [12,13,17,31,41,42].

Corresponding in the dual theory to the Brunn–Minkowski inequality (1) is the *dual Brunn–Minkowski inequality* for bounded Borel star sets C and D in $\mathbb{R}^n$, which states that

$$\mathcal{H}^n(C \mathbin{\widetilde{+}} D)^{1/n} \leqslant \mathcal{H}^n(C)^{1/n} + \mathcal{H}^n(D)^{1/n}, \tag{2}$$

where $\widetilde{+}$ denotes radial sum, with equality if and only if C is a dilatate of D. See, for example, [15, (B.30)] and [18, Section 3]. The reversal of the inequality sign in the passage from (1) to (2) is a standard, but not yet fully understood, feature of the duality at play.

Here we are interested in inequalities of the Brunn–Minkowski type for the p-capacity $\mathrm{Cap}_p(E)$ of a set E in $\mathbb{R}^n$, $1 \leqslant p < n$. The importance of p-capacity, $1 \leqslant p < n$ stems from the fact that if $E \subset \mathbb{R}^n$ and $\mathrm{Cap}_p(E) = 0$, then $\mathcal{H}^s(E) = 0$ for any $s > n-p$ (see [22, Theorem 2.26]), and in particular $\mathcal{H}^n(E) = 0$. Moreover, it turns out that Sobolev functions can be defined pointwise up to sets of capacity zero, and, as noted in [22, p. 27], capacity replaces measure in theorems for Sobolev functions of the Egorov and Lusin type. Thus p-capacity provides a finer scale of size than sets of measure zero in describing exceptional sets. In addition, to quote from [22, p. 27] again, capacity estimates play a decisive role in solutions to partial differential equations.

Our starting point is the fact that for convex bodies K and L in $\mathbb{R}^n$ and $1 \leqslant p < n$,

$$\mathrm{Cap}_p(K+L)^{1/(n-p)} \geqslant \mathrm{Cap}_p(K)^{1/(n-p)} + \mathrm{Cap}_p(L)^{1/(n-p)}, \tag{3}$$

with equality if and only if K and L are homothetic. This remarkable *capacitary Brunn–Minkowski inequality* was first proved by Borell [5] for electrostatic capacity (the case $p = 2$), with the equality condition established later by Caffarelli, Jerison, and Lieb [6]. (Both the inequality and equality condition were used by Jerison [24] in his solution of the corresponding

Minkowski problem.) A different proof and the extension to $1 < p < n$ was provided by Colesanti and Salani [8]. The case $p = 1$ follows from the fact that if K is a convex body in $\mathbb{R}^n$, then $\mathrm{Cap}_1(K)$ is just its surface area, $S(K)$ (see [29, Section 5.1]), and it is known that

$$S(K+L)^{1/n-1} \geqslant S(K)^{1/n-1} + S(L)^{1/n-1}, \tag{4}$$

with equality if and only if K and L are homothetic; see [14, (74), p. 393] and [36, (6.8.10), p. 385].

Note that the various set functions considered so far, as well as others mentioned below, are homogeneous. In particular, $\mathcal{H}^n(rE) = r^n\mathcal{H}^n(E)$, $\mathrm{Cap}_p(rE) = r^{n-p}\,\mathrm{Cap}_p(E)$ (see [9, Theorem 2(iv), p. 151]), and $S(rE) = r^{n-1}S(E)$, for $r > 0$ and appropriate sets E in $\mathbb{R}^n$. The exponents in the inequalities are therefore the natural ones and it automatically follows that equality holds when the sets concerned are dilatates of each other.

The principal conclusion of the present paper is that for star bodies C and D in $\mathbb{R}^n$, we have the *capacitary dual Brunn–Minkowski inequality*

$$\mathrm{Cap}_{n-1}(C \mathbin{\widetilde{+}} D) \leqslant \mathrm{Cap}_{n-1}(C) + \mathrm{Cap}_{n-1}(D), \tag{5}$$

with equality when $n = 2$ if and only if $\operatorname{conv} C$ is a dilatate of $\operatorname{conv} D$, and for bodies in $\mathbb{R}^n$, $n \geqslant 3$, that are star-shaped with respect to εB for some $\varepsilon > 0$ if and only if C is a dilatate of D. In fact the inequality itself for $n \geqslant 3$ follows directly from one proved by Bandle and Marcus [3] in the same year that the dual Brunn–Minkowski theory was born! Naturally the paper [3] makes no mention of the latter and the inequality (5) is somewhat hidden due to a quite different point of view and notation. Moreover, as for the capacitary Brunn–Minkowski inequality (3), much extra work (see Theorem 4.1) is required to establish an equality condition.

The inequality (5) for $n = 2$ is new (see Theorem 6.4), and it is interesting that its equality condition is different from that in higher dimensions. One step in proving this result is Lemma 5.1, which states that for Lipschitz star bodies C and D in $\mathbb{R}^2$,

$$S(C \mathbin{\widetilde{+}} D)^{1/(n-1)} \leqslant S(C)^{1/(n-1)} + S(D)^{1/(n-1)}. \tag{6}$$

In view of (4), it is natural to ask whether (6) is also true in higher dimensions, but it is a consequence of counterexamples constructed in Theorem 5.4 that (6) is *false* for $n \geqslant 3$.

Another natural question is whether (5) can be generalized to p-capacity for $1 \leqslant p < n$, that is, whether the dual counterpart to (3) is true. While we have no counterexample, in our opinion there is no particular reason to expect this to be the case. Indeed, consider the inequality

$$V_i(K+L)^{1/i} \geqslant V_i(K)^{1/i} + V_i(L)^{1/i}, \tag{7}$$

for the ith intrinsic volumes V_i, $1 \leqslant i \leqslant n$, of convex bodies K and L in $\mathbb{R}^n$ (of which the case $i = n$ when $V_n(K) = \mathcal{H}^n(K)$ is the Brunn–Minkowski inequality (1) and the case $i = n-1$ when $V_{n-1}(K) = (1/2)S(K)$ is (4); see [36, (6.8.10), p. 385]). The dual counterpart of (7) is the inequality

$$\widetilde{V}_i(C \mathbin{\widetilde{+}} D)^{1/i} \leqslant \widetilde{V}_i(C)^{1/i} + \widetilde{V}_i(D)^{1/i}, \tag{8}$$

for the ith dual volumes $\widetilde{V}_i$, $1 \leqslant i \leqslant n$, of Borel star sets C and D in $\mathbb{R}^n$ (of which the case $i = n$ when $\widetilde{V}_n(C) = \mathcal{H}^n(C)$ is the dual Brunn–Minkowski inequality (2); see [16, (50), p. 383]). However, it is only for $i = n$ that $V_i = \widetilde{V}_i$, and (8) is false in general, even when it makes sense, if $\widetilde{V}_i$ is replaced by V_i (as our Theorem 5.4 demonstrates in the case $i = n - 1$). In other words, while it is often the case that to an inequality in the classical Brunn–Minkowski theory there is a corresponding inequality in the dual Brunn–Minkowski theory, the set function concerned usually has to be replaced by a corresponding dual set function. Up to now, as far as we are aware, only volume $\mathcal{H}^n = V_n = \widetilde{V}_n$ was seen to be its own dual, appearing in both the classical inequality and its dual. It transpires from the case $p = n - 1$ of (3) and (5) that Cap_{n-1} is also its own dual and so is now seen to have the same status as volume in this sense.

The paper is organized as follows. After the preliminary Section 2, the notion of radial concentration is the focus of Section 3. In Section 4 we give a new proof of (5) when the dimension $n \geqslant 3$ and establish an equality condition in this case. The behavior of surface area under radial sums is examined in Section 5, and the final Section 6 is devoted to proving (5) and its equality condition when $n = 2$.

2. Definitions, notation, and preliminaries

As usual, S^{n-1} denotes the unit sphere, B the unit ball, o the origin, and $|\cdot|$ the norm in Euclidean n-space $\mathbb{R}^n$. If $x, y \in \mathbb{R}^n$, then $[x, y]$ denotes the line segment with endpoints x and y.

If X is a set, $\dim X$ is its *dimension*, that is, the dimension of its affine hull, ∂X is its *boundary*, $\operatorname{int} X$ its *interior* and $\operatorname{conv} X$ its *convex hull*. If $r > 0$, the set $rX = \{rx\colon x \in X\}$ is called a *dilatate* of X. If X and Y are sets in $\mathbb{R}^n$, then

$$X + Y = \{x + y\colon x \in X,\ y \in Y\}$$

is the *Minkowski* or *vector sum* of X and Y.

A *body* is a compact set equal to the closure of its interior. A *compact domain* is a connected body.

We write $\mathcal{H}^k$ for k-dimensional Hausdorff measure in $\mathbb{R}^n$, $k = 1, \dots, n$. If K is a k-dimensional body in $\mathbb{R}^n$, then we refer to $\mathcal{H}^k(K)$ as its *volume*. Define $\kappa_n = \mathcal{H}^n(B)$. The notation dz will always mean $d\mathcal{H}^k(z)$ for the appropriate $k = 1, \dots, n$.

If E is a measurable (i.e., $\mathcal{H}^n$-measurable) set in $\mathbb{R}^n$, we denote by $P(E)$ its *perimeter*. By [35, Theorem 1.8.2(1)], we can define $P(E) = \mathcal{H}^{n-1}(\partial_* E)$, where $\partial_* E$ is the essential boundary of E, the set of all points at which the upper Lebesgue density of E is positive and the lower Lebesgue density of E is less than one. If E is a body, we also refer to $P(E)$ as the *surface area* of E and denote it by $S(E)$.

A bounded set C is *star-shaped* at a point x if every line through x that meets C does so in a (possibly degenerate) closed line segment. If C is a set that contains the origin and is star-shaped at the origin, its *radial function* ρ_C is defined, for all $u \in S^{n-1}$, by

$$\rho_C(u) = \max\{c \geqslant 0\colon\ cu \in C\}.$$

In this paper, a *star set* is a set that contains the origin and is star-shaped at the origin.

By a *star body* in $\mathbb{R}^n$ we mean a body L star-shaped at the origin such that ρ_L is positive and continuous. Note that we are requiring that a star body contains the origin in its interior; more general definitions, such as that introduced in [19] (see also [15, Section 0.7]), allow bodies not containing the origin. A *Lipschitz star body* in $\mathbb{R}^n$ is a star body whose radial function is Lipschitz on S^{n-1} and hence (cf. [9, p. 81]) differentiable almost everywhere in S^{n-1}. For $\varepsilon > 0$, a set will be called *star-shaped with respect to* εB if it contains εB and is star-shaped at each $x \in \varepsilon B$. Note that by Lemma 4.3 below, such a set is a Lipschitz star body if it is bounded.

If $x, y \in \mathbb{R}^n$, then the *radial sum* $x \mathbin{\widetilde{+}} y$ of x and y is defined to be the usual vector sum $x + y$ if x and y are contained in a line through o, and o otherwise. If C and D are Borel star sets in $\mathbb{R}^n$ and $s, t \in \mathbb{R}$, then

$$sC \mathbin{\widetilde{+}} tD = \{sx \mathbin{\widetilde{+}} ty\colon\ x \in C,\ y \in D\},$$

and

$$\rho_{sC \widetilde{+} tD} = s\rho_C + t\rho_D.$$

A set in $\mathbb{R}^n$ is called a *convex body* if it is convex and compact with nonempty interior.

Minkowski's definition of surface area (see [15, (A.35), p. 405] or [40, p. 295]) allows $S(K)$ to be defined for compact convex sets in a way compatible with our earlier definition when K is a convex body. If $\dim K = n - 1$, however, Minkowski's definition gives $S(K) = 2\mathcal{H}^{n-1}(K)$ (note that in this case $P(K) = 0$!) and if $\dim K < n - 1$, then $S(K) = 0$.

The treatise of Schneider [36] is an excellent general reference for convex sets.

The *p-capacity* of an arbitrary set E in $\mathbb{R}^n$, for $1 \leqslant p < n$, is

$$\mathrm{Cap}_p(E) = \inf\left\{ \int_{\mathbb{R}^n} |\nabla f|^p \, dx \right\},$$

where the infimum is taken over all nonnegative functions f such that $f \in L^{np/(n-p)}(\mathbb{R}^n)$, $\nabla f \in L^p(\mathbb{R}^n; \mathbb{R}^n)$, and E is contained in the interior of $\{x\colon\ f(x) \geqslant 1\}$. See [9, p. 147]. We call the class of such functions f *admissible* for the p-capacity of E. The basic properties of p-capacity can be found in [9, p. 151] or [34, Section 2.2]. To these we add that if E is a Borel set, then

$$\mathrm{Cap}_p(E) = \sup\bigl\{\mathrm{Cap}_p(C)\colon\ C \subset E,\ C \text{ compact}\bigr\}. \tag{9}$$

This is a consequence of the monotonicity of Cap_p (see [9, p. 147]), [9, Theorem 2(i) and (viii), p. 151], and Choquet's capacitability theorem proved in [7].

For Cap_1, there is the following relation with $\mathcal{H}^{n-1}$:

Proposition 2.1. *For Borel sets E in $\mathbb{R}^n$, we have* $\mathrm{Cap}_1(E) = 0$ *if and only if* $\mathcal{H}^{n-1}(E) = 0$.

See [9, Theorem 3, p. 193], where the result is stated for compact sets; the more general statement above follows directly from (9) and the fact that (9) holds when Cap_p is replaced by $\mathcal{H}^{n-1}$ (see [10, Corollary 2.10.48]).

The following result can be found in the book by Maz'ya [34, Lemma 2.2.5].

Proposition 2.2. *If C is a compact subset of $\mathbb{R}^n$, then*

$$\mathrm{Cap}_1(C) = \inf\bigl\{S(E)\bigr\},$$

where the infimum is taken over all bodies E with C^∞ boundary that contain C.

Note that Proposition 2.2 is false in general for Borel sets. For example, the set of rationals in the unit cube in $\mathbb{R}^n$ has zero 1-capacity by Proposition 2.1 but any body containing this set must also contain the unit cube. In $\mathbb{R}^n$, $n \geqslant 3$, it is even false for connected Borel star sets. To see this, let D be a countable dense subset of S^{n-1}, and consider the set

$$C = \{(r, u) \in \mathbb{R}^n \colon 0 \leqslant r \leqslant 1,\ u \in D\},$$

where (r, u) denote polar coordinates. Then $\mathrm{Cap}_1(C) = 0$, again by Proposition 2.1, but any body containing C must also contain B.

Propositions 2.1 and 2.2 are essentially due to Fleming [11, Lemma 4.1 and Theorem 4.3]. For compact sets, the set function denoted γ by Fleming is just Cap_1. Note, however, that the extension of γ to arbitrary sets in [11, p. 457] is quite different from Cap_1.

3. Radial concentration and a generalization of radial sum

Let E be a bounded Borel set in $\mathbb{R}^n$. For each $u \in S^{n-1}$, let r_u be the half-infinite line (ray) emanating from the origin in the direction u. Define

$$E^* = \{x = (r, u) \colon 0 \leqslant r \leqslant \mathcal{H}^1(E \cap r_u)\}. \tag{10}$$

Here (r, u) are polar coordinates. In other words, E^* is the union of closed line segments with one endpoint at the origin, such that the length of each segment is the linear measure of the intersection of E with the ray in the same direction. In particular, E^* is a star set.

In using the notation E^* we are following Bandle and Marcus [2], and warn the reader that this notation is also commonly used for the polar body, a completely different concept. In [2], the authors call E^* the *radial concentration* of E; actually, we are only using a special case of their definition, corresponding to taking their function $g(r) \equiv 1$. On the other hand, our definition is more general in another sense, since in [2] the set E is required to be a bounded domain containing a ball centered at the origin. In this case E^* is a compact domain. More generally, it can be proved by standard arguments that if E is compact, then E^* is also compact.

For our more general definition, it is not immediately clear that since E is a Borel set, E^* is also a Borel set, but this can be proved using the argument in [16, Lemma 2.1]. (To see this, note that in the notation of [16], $\widetilde{\triangle} E$, the chordal symmetral of E, is defined by

$$\widetilde{\triangle} E = \{x = (r, u) \colon 0 \leqslant r \leqslant \mathcal{H}^1(E \cap l_u)/2\},$$

where l_u is the line through the origin parallel to u. Then the proof of [16, Lemma 2.1] is easily adapted.) As an aside, we mention that E^* was called the *directed chordal symmetral* of E at o by Gardner [15, p. 198]. Also, if K is a convex body in $\mathbb{R}^n$, then by a result of Longinetti (see [15, Theorem 5.1.5]), K^* is also a convex body. The more general fact that if K is any bounded convex set in $\mathbb{R}^n$, then K^* is a compact convex set is an easy consequence of this and (10).

Let C and D be bounded Borel sets in $\mathbb{R}^n$. We define the *radial sum* of C and D by

$$C \mathbin{\widetilde{+}} D = C^* \mathbin{\widetilde{+}} D^*, \tag{11}$$

where the right-hand side makes sense because C^* and D^* are Borel star sets. Note that this more general definition of radial sum is compatible with the earlier one, since $E^* = E$ for Borel star sets E.

Since C^* and D^* are Borel sets, $C \,\widetilde{+}\, D$ is also a Borel set. To see this, note that ρ_{C^*} and ρ_{D^*} are Borel functions, so $\rho_{C^* \widetilde{+} D^*}$ is also a Borel function. It then follows from [23, Lemma 11.9] that $C^* \,\widetilde{+}\, D^*$ is a Borel set.

It is also not difficult to show that if C and D are compact sets, then $C \,\widetilde{+}\, D$ is compact. Unfortunately $C \,\widetilde{+}\, D$ need not be a body even when C and D are convex bodies. For example, let $C = B \cap \{(x, y) \in \mathbb{R}^2\colon\ y \geqslant 0\}$ and $D = B \cap \{(x, y) \in \mathbb{R}^2\colon\ y \leqslant 0\}$. Then $C \,\widetilde{+}\, D = B \cup L$, where $L = [(-2, 0), (2, 0)]$.

4. A capacitary dual Brunn–Minkowski inequality with equality condition

The inequality in the following theorem follows from a result of Bandle and Marcus [3, Theorem 3.1]. However, it does not seem possible to obtain any equality condition by the method employed there. To prove the new equality condition stated, we shall have to obtain the inequality, under some extra assumptions, by a quite different (though overlapping) method. In fact, underlying both proofs is a level-set change-of-variables technique introduced by Szegö [38].

Theorem 4.1. *Let C and D be compact domains in $\mathbb{R}^n$, $n \geqslant 3$. Then*

$$\mathrm{Cap}_{n-1}(C \,\widetilde{+}\, D) \leqslant \mathrm{Cap}_{n-1}(C) + \mathrm{Cap}_{n-1}(D). \tag{12}$$

If C and D are bodies in $\mathbb{R}^n$ that are star-shaped with respect to εB for some $\varepsilon > 0$, then equality holds if and only if C is a dilatate of D.

We shall need some preliminary facts and lemmas. Material on the Sobolev spaces $W^{1,p}(G)$ and $W_0^{1,p}(G)$, where G is an open set, can be found, for example, in [1] and [20, Chapter 7]. For $1 < p < \infty$, we use the standard notation

$$\Delta_p u = \mathrm{div}\big(|\nabla u|^{p-2} \nabla u\big)$$

for the p-Laplacian of u.

Let C and Ω be compact domains in $\mathbb{R}^n$ with $C \subset \mathrm{int}\,\Omega$, and let $1 < p < n$. Consider the problem

$$\begin{cases} \Delta_p u = 0 \text{ in } \mathrm{int}\,\Omega \setminus C, \\ u = 1 \text{ on } \partial C, \quad u = 0 \text{ on } \partial\Omega. \end{cases} \tag{13}$$

If C and Ω are Lipschitz compact domains, it is known that there is a unique function $u_C^\Omega \in W_0^{1,p}(\mathrm{int}\,\Omega) \cap C(\Omega) \cap C^1(\mathrm{int}\,\Omega \setminus C)$ such that u_C^Ω is a weak solution of (13), $0 \leqslant u_C^\Omega(x) \leqslant 1$ for all $x \in \Omega$, and $u_C^\Omega(x) = 1$ for $x \in C$. If $\Omega = RB$, we shall write u_C^R instead of u_C^{RB}.

Probably most of these facts are known to many readers, but we include references for completeness. The proofs of the existence of a weak solution to (13) in $W^{1,p}(\mathrm{int}\,\Omega \setminus C)$ and the continuity of the solution in $\mathrm{int}\,\Omega \setminus C$ are sketched in [27, pp. 202, 203]. The fact that a solution is continuous up to the boundary of $\Omega \setminus C$ follows from the fact that C and Ω are Lipschitz, and hence the domain satisfies an exterior cone condition (see [22, pp. 122, 123]). Extending a solution to be 1 in C gives us a continuous function in Ω. A further consequence of C being Lipschitz, the continuity of the solution, and the fact that it vanishes on $\partial\Omega$, is that this extension belongs to $W_0^{1,p}(\operatorname{int}\Omega)$. The uniqueness is a consequence of the following maximum principle for p-harmonic functions (see, for example, [27, Lemma 1, p. 206]), which we shall also make use of later.

Proposition 4.2. *Let $1 < p < \infty$. Suppose v and w are continuous functions on $\mathbb{R}^n$ that are weak solutions of $\Delta_p u = 0$ in a bounded open set G and satisfy $v \leqslant w$ on ∂G. Then $v \leqslant w$ in G.*

We have $u_C^\Omega \in C^1(\operatorname{int}\Omega \setminus C)$ by [28, Theorem 1]. Finally, the fact that $0 < u_C^\Omega < 1$ in $\operatorname{int}\Omega \setminus C$ is a consequence of the strong maximum principle for C^1 p-harmonic functions; see [39, Theorem 5].

Now consider the problem

$$\begin{cases} \Delta_p u = 0 \text{ in } \mathbb{R}^n \setminus C, \\ u = 1 \text{ on } \partial C, \quad \lim_{|x|\to\infty} u(x) = 0. \end{cases} \tag{14}$$

If C is a Lipschitz compact domain in $\mathbb{R}^n$, it is known that there is a unique function $u_C \in C(\mathbb{R}^n) \cap C^1(\mathbb{R}^n \setminus C)$ such that u_C is a weak solution of (14), $0 < u_C(x) \leqslant 1$ for all $x \in \mathbb{R}^n$, $u_C(x) = 1$ for $x \in C$, and

$$\operatorname{Cap}_p(C) = \int_{\mathbb{R}^n} |\nabla u_C|^p \, dx. \tag{15}$$

In fact $u_C^R \to u_C$ as $R \to \infty$ in the C^1 sense on compact subsets of $\mathbb{R}^n \setminus C$. The function u_C is called the *p-capacitary function* of C. The proof of these facts is the same as that of [8, Theorem 2]; the convexity assumed there is only needed for additional conclusions.

Lemma 4.3. *Let $\varepsilon, R > 0$ and let C be a body in $\mathbb{R}^n$ contained in* $\operatorname{int}(RB)$ *and star-shaped with respect to εB. Then C is a Lipschitz star body with Lipschitz constant depending only on ε and R.*

Proof. Let C be as in the statement of the lemma and let $x \in \partial C$. Let r_x be the ray emanating from o and passing through x, and let $u = x/|x|$. Because C is star-shaped with respect to εB, x is the unique point in $\partial C \cap r_x$. The set $\operatorname{conv}(\varepsilon B \cup \{x\})$ is contained in C and is the union of εB and a finite cone with vertex at x. Let $0 < \alpha < \pi/2$ be the vertex half-angle of this cone. Then the infinite cone Q with vertex x, axis along r_x, vertex half-angle α and disjoint from εB does not meet $\operatorname{int} C$.

Let p be the point where r_x meets $\partial(RB)$. The set $\operatorname{conv}(\varepsilon B \cup \{p\})$ is the union of εB and a finite cone with vertex at p. Let α_1 be the vertex half-angle of this cone. Then $0 < \alpha_1 < \alpha = \alpha(x)$. Let $v \in S^{n-1}$ be such that the angle θ between u and v is no greater than $\alpha_1/2$, and let r_v be the ray emanating from o and passing through v. Then there are unique points y and z in $\partial(\operatorname{conv}(\varepsilon B \cup \{x\})) \cap r_v$ and $\partial Q \cap r_v$, respectively. Clearly

$$\bigl|\rho_C(u) - \rho_C(v)\bigr| \leqslant \max\bigl\{\bigl||x| - |y|\bigr|, \bigl||x| - |z|\bigr|\bigr\} \leqslant \max\bigl\{|x - y|, |x - z|\bigr\}.$$

By the law of sines we have

$$|u - v| = \frac{\sin\theta}{\sin(\pi - \theta)/2} \tag{16}$$

and hence

$$|x - y| = \frac{|y|\sin\theta}{\sin\alpha} = \frac{|y||u - v|\sin(\pi - \theta)/2}{\sin\alpha} < \frac{R|u - v|}{\sin\alpha_1}.$$

Similarly,

$$|x - z| = \frac{|x|\sin\theta}{\sin\beta} < \frac{R|u - v|}{\sin(\alpha_1/2)},$$

where $\beta \geqslant \alpha_1/2$ is the angle at z of the triangle with vertices 0, x, and z. Consequently,

$$\left|\rho_C(u) - \rho_C(v)\right| < \frac{R|u - v|}{\sin(\alpha_1/2)},$$

for all $v \in S^{n-1}$ such that the angle θ between u and v is no greater than $\alpha_1/2$. Now suppose that $v \in S^{n-1}$ is such that $\theta > \alpha_1/2$. By (16) we have $|u - v| > \sin(\alpha_1/2)$, so

$$\left|\rho_C(u) - \rho_C(v)\right| < 2R < \frac{2R|u - v|}{\sin(\alpha_1/2)}.$$

Since α_1 depends only on ε and R, the proof is complete. $\square$

Lemma 4.4. *Let $\varepsilon, R > 0$ and let C be a body in $\mathbb{R}^n$ contained in* $\mathrm{int}(RB)$ *and star-shaped with respect to εB. Let $1 < p < n$ and let u_C^R and u_C be the solutions of* (13) *and* (14)*, respectively. Then for all $0 < t \leqslant 1$, the t-superlevel sets*

$$C^{R,t} = \left\{x \in \mathbb{R}^n\colon\ u_C^R(x) \geqslant t\right\} \quad \text{and} \quad C^t = \left\{x \in \mathbb{R}^n\colon\ u_C(x) \geqslant t\right\}$$

are star-shaped with respect to εB. In particular, $C^{R,1} = C^1 = C$.

Proof. Let $0 < t < 1$. By [25, Corollary 1.2], $C^{R,t}$ is a star set. If there exists a C^t that is not star-shaped, there must be a ray r emanating from the origin such that there is a bounded component A of $r \setminus C^t$. By the continuity of u_C, there exists $\delta > 0$ such that $A \cap \{x \in \mathbb{R}^n\colon\ u_C(x) < t - \delta\}$ is nonempty. By the uniform convergence of u_C^R to u_C on compact subsets, there exists R_0 such that if $R > R_0$, then $C^t \subset C^{R,t-\delta}$. Because $C^{R,t-\delta}$ is star-shaped, it must contain A. Using the uniform convergence again, this implies that $u_C \geqslant t - \delta$ on A, a contradiction. Therefore C^t is a star set.

Now let $x_0 \in \varepsilon B$ and let $\bar{u}_C^R(x) = u_C^R(x + x_0)$ for all $x \in RB - x_0$. It is easy to check that $\bar{u}_C^R$ is a weak solution of (13) when C and Ω are replaced by $C - x_0$ and $RB - x_0$, respectively. Because C and RB are star-shaped at x_0, the bodies $C - x_0$ and $RB - x_0$ are star sets. By [25, Corollary 1.2] again, the set

$$\{x \in \mathbb{R}^n: \bar{u}_C^R(x) \geqslant t\} = \{x \in \mathbb{R}^n: u_C^R(x + x_0) \geqslant t\} = C^{R,t} - x_0$$

is a star set. Hence $C^{R,t}$ is star-shaped at x_0, and this shows that $C^{R,t}$ is star-shaped with respect to εB. An analogous argument shows that C^t is star-shaped with respect to εB for $0 < t < 1$.

By the strong maximum principle [39, Theorem 5] applied to $1 - u_C^R$ in $RB \setminus C$, $u_C^R < 1$ in $RB \setminus C$, so that $C^{R,1} = C$. Similarly $C^1 = C$. □

Let $\varepsilon, R > 0$ and let C and D be bodies in $\mathbb{R}^n$ contained in $\operatorname{int}(RB)$ and star-shaped with respect to εB. For $0 < t \leqslant 1$, let $C^{R,t}$, $D^{R,t}$, C^t, and D^t denote the t-superlevel sets of u_C^R, u_D^R, u_C, and u_D, respectively. By the previous lemma, the radial functions $\rho_{C^{R,t}}$, $\rho_{D^{R,t}}$, ρ_{C^t}, and ρ_{D^t} of these sets are well defined, and moreover, by Lemmas 4.3 and 4.4 and the fact that C^t is bounded, these are all Lipschitz functions on S^{n-1}. Let w^R be the function on $2RB$ such that $w^R(x) = 1$ for all $x \in C \mathbin{\widetilde{+}} D$, $w^R(x) = 0$ for $x \in \partial(2RB)$, and for $0 < t \leqslant 1$, the t-superlevel set

$$E^{R,t} = \{x \in \mathbb{R}^n: w^R(x) \geqslant t\} \tag{17}$$

of w^R is given by

$$\rho_{E^{R,t}} = \rho_{C^{R,t}} + \rho_{D^{R,t}}. \tag{18}$$

Similarly, let w be the function on $\mathbb{R}^n$ such that $w(x) = 1$ for all $x \in C \mathbin{\widetilde{+}} D$ and for $0 < t \leqslant 1$, the t-superlevel set

$$E^t = \{x \in \mathbb{R}^n: w(x) \geqslant t\} \tag{19}$$

of w is given by

$$\rho_{E^t} = \rho_{C^t} + \rho_{D^t}. \tag{20}$$

Equivalently, the functions w^R and w can be defined by

$$w^R(x) = \sup\{t: x \in C^{R,t} \mathbin{\widetilde{+}} D^{R,t}\} \tag{21}$$

and

$$w(x) = \sup\{t: x \in C^t \mathbin{\widetilde{+}} D^t\}. \tag{22}$$

Lemma 4.5. *Let $\varepsilon, R > 0$ and let C and D be bodies in $\mathbb{R}^n$ contained in* $\operatorname{int}(RB)$ *and star-shaped with respect to εB. Let w^R be defined by* (17) *and* (18) *and let w be defined by* (19) *and* (20). *Then w^R is Lipschitz on compact subsets of* $\operatorname{int}(2RB) \setminus (C \mathbin{\widetilde{+}} D)$ *and w is Lipschitz on compact subsets of* $\mathbb{R}^n \setminus (C \mathbin{\widetilde{+}} D)$.

Proof. The claim about w^R appears without proof in [3, Section 3.2], where the authors refer to [32]. However, the latter paper deals with a different combination of the radial functions of C and D, and only in two dimensions. For this reason we provide the following self-contained proof.

Let F be a compact subset of $\operatorname{int}(2RB) \setminus (C \mathbin{\widetilde{+}} D)$, and let $x, y \in F$. We need only consider the case $|x-y| < \delta$, where $0 < \delta < \min\{\varepsilon, 1\}$ is to be chosen, since if $|x-y| \geqslant \delta$,

$$\left|w^R(x) - w^R(y)\right| \leqslant 1 \leqslant (1/\delta)|x-y|.$$

Let r_x and r_y be the rays emanating from o and containing x and y, respectively, and let θ be the angle between r_x and r_y. By choosing δ small enough (depending only on F), we can guarantee that θ is as small as we like.

Let $t_1 = w^R(x)$ and $t_2 = w^R(y)$, and assume that $t_1 \geqslant t_2$. By Lemma 4.4 applied to C^{R,t_1}, C^{R,t_2}, D^{R,t_1}, and D^{R,t_2}, there exist unique points x_C and x_D on r_x and unique points y_C and y_D on r_y such that $x = x_C + x_D$, $y = y_C + y_D$, $u_C^R(x_C) = u_D^R(x_D) = t_1$, and $u_C^R(y_C) = u_D^R(y_D) = t_2$. Note also that $|x| = |x_C| + |x_D|$ and $|y| = |y_C| + |y_D|$.

Define

$$F_C^R = \left\{x \in RB \setminus C\colon \inf_F w^R \leqslant u_C^R(x) \leqslant \sup_F w^R\right\}, \tag{23}$$

and define F_D^R analogously. Because w^R is bounded away from 0 and 1 in F, F_C^R and F_D^R are compact subsets of $\operatorname{int}(RB) \setminus C$ and $\operatorname{int}(RB) \setminus D$, respectively. Note also that $x_C, y_C \in F_C^R$ and $x_D, y_D \in F_D^R$. Since $u_D^R \in C^1(\operatorname{int}(RB) \setminus D)$, it is Lipschitz on F_D^R with Lipschitz constant c_0, say. Then

$$\left|w^R(x) - w^R(y)\right| = t_1 - t_2 = \left|u_D^R(x_D) - u_D^R(y_D)\right| \leqslant c_0|x_D - y_D|.$$

Therefore it suffices to prove that

$$|x_D - y_D| \leqslant c_1|x-y| \tag{24}$$

for some constant c_1 independent of x and y. If $r_x = r_y$, we have $|x_C| \leqslant |y_C|$ and $|x_D| \leqslant |y_D|$, so $|x_C - y_C| + |x_D - y_D| = |x-y|$ and (24) holds with $c_1 = 1$. Thus we may assume that $r_x \neq r_y$.

By Lemma 4.3, $\rho_{C^{R,t_1}}$ is Lipschitz with Lipschitz constant, c_2 say, depending only on ε and the diameter of F_C^R. Therefore,

$$|x_C| - \rho_{C^{R,t_1}}\bigl(y/|y|\bigr) = \rho_{C^{R,t_1}}\bigl(x/|x|\bigr) - \rho_{C^{R,t_1}}\bigl(y/|y|\bigr) \leqslant c_2\theta.$$

Now because $t_2 = u_C^R(y_C) \leqslant t_1$ and u_C^R decreases on r_y, we have $|y_C| \geqslant \rho_{C^{R,t_1}}(y/|y|)$. Therefore

$$|x_C| - |y_C| \leqslant c_2\theta, \tag{25}$$

and hence

$$|y_D| - |x_D| = |y| - |y_C| - |x| + |x_C| \leqslant |x-y| + c_2\theta. \tag{26}$$

Just as we obtained (25), we obtain also that

$$|x_D| - |y_D| \leqslant c_3\theta, \tag{27}$$

where c_3 is a constant depending only ε and the diameter of F_D^R. Combining (26) and (27), we have

$$\big||x_D| - |y_D|\big| \leqslant |x - y| + c_4\theta$$

where $c_4 = \max\{c_2, c_3\}$. From the fact that $|x|, |y| \geqslant \varepsilon$, it is easy to see that $\theta \leqslant c_5|x - y|$, where c_5 depends only on ε, and therefore

$$\big||x_D| - |y_D|\big| \leqslant c_6|x - y|$$

where c_6 depends only on ε and the diameters of F_C^R and F_D^R. Let $z \in r_y$ be such that $|z| = |x_D|$. Then

$$|x_D - y_D| \leqslant |x_D - z| + |z - y_D| \leqslant |x_D|\theta + \big||x_D| - |y_D|\big| \leqslant \big(c_5|x_D| + c_6\big)|x - y| = c_7|x - y|,$$

where, since $|x_D| < |x|$, c_7 depends on F, ε, and the diameters of F_C^R and F_D^R. This establishes (24) with $c_1 = c_7$ and completes the proof that w^R is Lipschitz in F.

To prove the statement about w, we first show that the Lipschitz constants of w^R are uniformly bounded in each compact subset F of $\mathbb{R}^n \setminus (C \mathbin{\widetilde{+}} D)$. Suppose that $F \subset \operatorname{int}(2RB)$ for $R \geqslant R_0$, say. We have shown that the Lipschitz constant for w^R in F is controlled by a constant depending on F, ε, the Lipschitz constant of u_D^R in F_D^R, and the diameters of the sets F_C^R and F_D^R, so it suffices to prove that the Lipschitz constant of u_D^R in F_D^R and the diameters of F_C^R and F_D^R can be bounded independently of R.

To this end, note that w is bounded away from 1 in F. Because F is a compact subset of $\operatorname{int}(2R_0B)$, there exists $\varepsilon_0 > 0$ such that $w^{R_0}(x) \geqslant \varepsilon_0$ for all $x \in F$. Define

$$F_C = \Big\{x \in \mathbb{R}^n \setminus C\colon\ \varepsilon_0 \leqslant u_C(x) \text{ and } u_C^{R_0}(x) \leqslant \sup_F w\Big\}$$

and define F_D similarly. Suppose $R_1 < R_2$. Because $u_C^{R_2} > 0$ on $\partial(R_1B)$, Proposition 4.2 implies that $u_C^{R_1}(x) \leqslant u_C^{R_2}(x)$ for any $x \in R_1B$, and therefore $u_C^R(x) \leqslant u_C(x)$ for all $x \in RB$ and all $R \geqslant R_0$. The corresponding statements also hold for the functions u_D^R and u_D. Consequently by (21) and (22), $w^{R_1}(x) \leqslant w^{R_2}(x)$ for any $x \in 2R_1B$ and $w^R(x) \leqslant w(x)$ for any $x \in 2RB$. Now if $x \in F_C^R$ for some $R \geqslant R_0$, then by (23), $u_C^{R_0}(x) \leqslant u_C^R(x) \leqslant \sup_F w^R \leqslant \sup_F w$, and $u_C(x) \geqslant u_C^R(x) \geqslant \inf_F w^R \geqslant \inf_F w^{R_0} \geqslant \varepsilon_0$, so $x \in F_C$. Therefore $F_C^R \subset F_C$ and similarly $F_D^R \subset F_D$ for all $R \geqslant R_0$. It follows that the diameters of F_C^R and F_D^R are bounded by the diameters of F_C and F_D, respectively. Moreover, the Lipschitz constant of u_D^R in F_D^R is no larger than the Lipschitz constant of u_D^R in F_D. Since $u_D^R \to u_D$ as $R \to \infty$ in the C^1 sense on compact subsets of $\mathbb{R}^n \setminus D$, $|\nabla u_D^R|$ is uniformly bounded on each compact subset of $\mathbb{R}^n \setminus D$, which implies a uniform Lipschitz bound on u_D^R in F_D. We conclude that the Lipschitz constants of w^R are uniformly bounded in F.

We now claim that w^R converges uniformly on compact subsets of $\mathbb{R}^n \setminus (C \mathbin{\widetilde{+}} D)$ to w as $R \to \infty$. From this and the bound on the Lipschitz constants of w^R just established it follows that w is Lipschitz on compact subsets of $\mathbb{R}^n \setminus (C \mathbin{\widetilde{+}} D)$ and the proof is complete.

To prove the claim, let F be a compact subset of $\mathbb{R}^n \setminus (C \mathbin{\widetilde{+}} D)$. Then F must be contained in a set of the form $\{x\colon\ a \leqslant w(x) \leqslant b\}$ for some $a > 0$ and $b < 1$. Let $x \in F$ and let $t = w(x)$.

By Lemma 4.4, there exist unique points $x_C \in \partial C^t$ and $x_D \in \partial D^t$ such that $x = x_C \tilde{+} x_D$. Note that $x_C \in \{y\colon a \leqslant u_C(y) \leqslant b\}$ and $x_D \in \{y\colon a \leqslant u_D(y) \leqslant b\}$, and that these are compact subsets of $\mathbb{R}^n \setminus C$ and $\mathbb{R}^n \setminus D$, respectively. Let $\varepsilon_1 > 0$. By the uniform convergence on compact sets of u_C^R to u_C and of u_D^R to u_D, there exists an R_3, independent of x, such that $u_C^R(x_C) > t - \varepsilon_1$ and $u_D^R(x_D) > t - \varepsilon_1$ for all $R \geqslant R_3$. Then, recalling that $w \geqslant w^R$ and using (21), we obtain

$$0 \leqslant w(x) - w^R(x) \leqslant t - (t - \varepsilon_1) = \varepsilon_1,$$

for all $R \geqslant R_3$. This proves the claim. $\square$

The following algebraic lemma is crucial in our proof (as it was in [3]).

Lemma 4.6. *For $x_0, y_0 > 0$, and reals x_i, y_i, $i = 1, \ldots, n$, we have*

$$\frac{(\sum_{i=1}^n (x_i + y_i)^2)^{(n-1)/2}}{(x_0 + y_0)^{n-2}} \leqslant \frac{(\sum_{i=1}^n x_i^2)^{(n-1)/2}}{x_0^{n-2}} + \frac{(\sum_{i=1}^n y_i^2)^{(n-1)/2}}{y_0^{n-2}}, \tag{28}$$

with equality if and only if either $x_i = y_i = 0$ for $i = 1, \ldots, n$ or $x_i = \alpha y_i$, $i = 0, \ldots, n$, for some $\alpha > 0$.

Proof. By the triangle inequality and Hölder's inequality with exponents $p = n - 1$ and $q = (n-1)/(n-2)$, we obtain

$$\begin{aligned}
\left(\sum_{i=1}^n (x_i + y_i)^2\right)^{1/2} &\leqslant \left(\sum_{i=1}^n x_i^2\right)^{1/2} + \left(\sum_{i=1}^n y_i^2\right)^{1/2} \\
&= \left(\frac{(\sum_{i=1}^n x_i^2)^{1/2}}{x_0^{(n-2)/(n-1)}}\right) x_0^{(n-2)/(n-1)} + \left(\frac{(\sum_{i=1}^n y_i^2)^{1/2}}{y_0^{(n-2)/(n-1)}}\right) y_0^{(n-2)/(n-1)} \\
&\leqslant \left(\left(\frac{(\sum_{i=1}^n x_i^2)^{1/2}}{x_0^{(n-2)/(n-1)}}\right)^{n-1} + \left(\frac{(\sum_{i=1}^n y_i^2)^{1/2}}{y_0^{(n-2)/(n-1)}}\right)^{n-1}\right)^{1/(n-1)} \\
&\quad \times \left(\left(x_0^{(n-2)/(n-1)}\right)^{(n-1)/(n-2)} + \left(y_0^{(n-2)/(n-1)}\right)^{(n-1)/(n-2)}\right)^{(n-2)/(n-1)} \\
&= \left(\frac{(\sum_{i=1}^n x_i^2)^{(n-1)/2}}{x_0^{n-2}} + \frac{(\sum_{i=1}^n y_i^2)^{(n-1)/2}}{y_0^{n-2}}\right)^{1/(n-1)} (x_0 + y_0)^{(n-2)/(n-1)}.
\end{aligned}$$

Rearranging, we get (28).

Suppose that equality holds in (28). Then equality holds in the triangle inequality, which implies that $x_i = \alpha y_i$ for $i = 1, \ldots, n$ and some $\alpha \geqslant 0$. Equality also holds in Hölder's inequality, implying that there are constants β and γ with $\beta^2 + \gamma^2 > 0$ such that

$$\beta\left(\frac{(\sum_{i=1}^n x_i^2)^{1/2}}{x_0^{(n-2)/(n-1)}}\right)^{n-1} = \gamma\left(x_0^{(n-2)/(n-1)}\right)^{(n-1)/(n-2)},$$

or equivalently

$$\beta\left(\sum_{i=1}^{n} x_i^2\right)^{(n-1)/2} = \gamma x_0^{n-1},$$

and the same equation with y_i instead of x_i, $i = 0, \ldots, n$. Therefore

$$\gamma x_0^{n-1} = \beta\left(\sum_{i=1}^{n} x_i^2\right)^{(n-1)/2} = \beta\left(\sum_{i=1}^{n} (\alpha y_i)^2\right)^{(n-1)/2} = \gamma(\alpha y_0)^{n-1}.$$

If $\gamma = 0$, then $x_i = y_i = 0$ for $i = 1, \ldots, n$. If $\gamma \neq 0$, then $\alpha > 0$ and $x_i = \alpha y_i$ for $i = 0, \ldots, n$. $\square$

Lemma 4.7. *Let $\varepsilon, R > 0$ and let C and D be bodies in $\mathbb{R}^n$, $n \geqslant 3$, contained in* $\mathrm{int}(RB)$ *and star-shaped with respect to εB. Let w^R be defined by* (17) *and* (18)*, and let w be defined by* (19) *and* (20)*. Then*

$$\int_{2RB} \left|\nabla w^R\right|^{n-1} dx \leqslant \int_{RB} \left|\nabla u_C^R\right|^{n-1} dx + \int_{RB} \left|\nabla u_D^R\right|^{n-1} dx \tag{29}$$

and

$$\int_{\mathbb{R}^n} |\nabla w|^{n-1} dx \leqslant \int_{\mathbb{R}^n} |\nabla u_C|^{n-1} dx + \int_{\mathbb{R}^n} |\nabla u_D|^{n-1} dx. \tag{30}$$

Proof. Let (ρ, θ), $\theta = (\theta_1, \ldots, \theta_{n-1})$, denote spherical polar coordinates in $\mathbb{R}^n$. By Lemmas 4.3 and 4.4, the functions $\rho_{C^{R,t}}$ and $\rho_{D^{R,t}}$ are Lipschitz on S^{n-1} for each $t \in [0, 1]$. The same is true of $\rho_{E^{R,t}}$ by (18). Therefore, by [9, p. 81], $(\rho_{C^{R,t}})_{\theta_i}(\theta)$, $(\rho_{D^{R,t}})_{\theta_i}(\theta)$, and $(\rho_{E^{R,t}})_{\theta_i}(\theta)$ exist for almost all t, θ, and $1 \leqslant i \leqslant n-1$. Because the level sets $C^{R,t}$ and $D^{R,t}$ are star-shaped with respect to εB and decrease as t increases, the functions u_C^R and u_D^R are strictly decreasing along rays emanating from the origin. This implies that $\rho_{C^{R,t}}(\theta)$ and $\rho_{D^{R,t}}(\theta)$ are strictly decreasing functions of t for each θ, and so by (18) again, so is $\rho_{E^{R,t}}$. Thus for each θ, $(\rho_{C^{R,t}})_t(\theta)$, $(\rho_{D^{R,t}})_t(\theta)$, and $(\rho_{E^{R,t}})_t(\theta)$ exist for almost all t.

Let $x \in \partial C^{R,t}$. Since $\rho_{C^{R,t}}$ is Lipschitz on S^{n-1}, $\partial C^{R,t}$ cannot be tangential to the ray emanating from the origin and passing through x. It follows that if

$$\frac{\partial u_C^R}{\partial \rho}(x) = \nabla u_C^R(x) \cdot \frac{x}{|x|} = 0,$$

then $|\nabla u_C^R(x)| = 0$, and the analogous statement is true for u_D^R. Since $\rho_{E^{R,t}}$ is Lipschitz on S^{n-1} by (18), the analogous statement is also true for w^R at all $x \in \mathrm{int}(2RB) \setminus (C \mathbin{\widetilde{+}} D)$ where w^R is differentiable.

Let $U_C = \{x \in \mathrm{int}(RB) \setminus C\colon |\nabla u_C^R(x)| \neq 0\}$. Then U_C is an open set on which u_C^R is not only strictly decreasing along each ray emanating from the origin, but also has a nonzero directional derivative in the direction of the ray. Let

$$V_C = \left\{(t, \theta) \in (0, 1) \times S^{n-1}\colon \left|\nabla u_C^R\left(\rho_{C^{R,t}}(\theta), \theta\right)\right| \neq 0\right\} = \left\{(t, \theta)\colon \left(\rho_{C^{R,t}}(\theta), \theta\right) \in U_C\right\}.$$

Define U_D and V_D analogously, by replacing C by D.

In the next part of the proof, it will be convenient to use the simpler notations $(u, U, V, F(t,\theta))$ for $(u_C^R, U_C, V_C, \rho_{C^{R,t}}(\theta))$ or $(u_D^R, U_D, V_D, \rho_{D^{R,t}}(\theta))$, where $0 < t < 1$ and $\theta \in S^{n-1}$. We have

$$|\nabla u|^2 = u_\rho^2 + \sum_{i=1}^{n-1} \frac{k_i(\theta)}{\rho^2} u_{\theta_i}^2, \tag{31}$$

where $k_i(\theta)$ is a nonnegative function (the square of some function). We also have

$$dx = \rho^{n-1} j(\theta)\, d\rho\, d\theta_1 \dots d\theta_{n-1} \tag{32}$$

for some nonnegative function $j(\theta)$. By definition,

$$u\big(F(t,\theta), \theta_1, \dots, \theta_{n-1}\big) = t. \tag{33}$$

The preceding discussion shows that for almost all (t,θ) we may differentiate (33) to obtain

$$u_\rho F_t = 1 \quad \text{and} \quad u_\rho F_{\theta_i} + u_{\theta_i} = 0, \quad i = 1, \dots, n-1, \tag{34}$$

which gives

$$u_{\theta_i} = -\frac{F_{\theta_i}}{F_t}, \quad i = 1, \dots, n-1. \tag{35}$$

By (31) and (32), we have

$$\begin{aligned} \int\limits_{RB\setminus\{u=1\}} |\nabla u|^{n-1}\, dx &= \int\limits_U |\nabla u|^{n-1}\, dx \\ &= \int\limits_{S^{n-1}} \int\limits_{U(\theta)} \left(u_\rho^2 + \sum_{i=1}^{n-1} \frac{k_i(\theta)}{\rho^2} u_{\theta_i}^2 \right)^{(n-1)/2} \rho^{n-1}\, d\rho\, j(\theta)\, d\theta, \end{aligned} \tag{36}$$

where we have written $U(\theta) = \{\rho\colon\ (\rho,\theta) \in U\} \subset (F(1,\theta), R)$. Note that the inner integral is finite for almost every θ, because $u \in W^{1,n-1}(\mathrm{int}(RB) \setminus \{u = 1\})$.

The next step in the argument is to change the variable in the inner integral in (36), via the formulas (34) and (35). For each θ, $U(\theta)$ is open in $(F(1,\theta), R)$ and $g(\rho) = u(\rho\theta)$ is C^1 and satisfies $g'(\rho) < 0$ on $U(\theta)$. Let $U(\theta) = \bigcup_{j=1}^{\infty} (a_j, b_j)$, where the open intervals (a_j, b_j) are disjoint. By (33), $f(t) = F(t,\theta)$ for $t \in [0,1]$ is the inverse of g, and f is C^1 on $V(\theta) = \{t \in (0,1)\colon\ (t,\theta) \in V\}$. Let $V(\theta) = \bigcup_{j=1}^{\infty} (c_j, d_j)$ where $f(c_j) = b_j$ and $f(d_j) = a_j$. Since $|f'|$ is bounded on each compact subset of (c_j, d_j), f is locally absolutely continuous on each (c_j, d_j). By [37, Corollary 6 and p. 519], we can apply the change of variables defined by (33), using (34), (35), and the fact that $F_t < 0$, to obtain

$$\int_{a_j}^{b_j} \left(u_\rho^2 + \sum_{i=1}^{n-1} \frac{k_i(\theta)}{\rho^2} u_{\theta_i}^2 \right)^{(n-1)/2} \rho^{n-1}\, d\rho = \int_{c_j}^{d_j} \left(\frac{1}{(F_t)^2} + \sum_{i=1}^{n-1} \frac{k_i(\theta)}{F^2} \left(\frac{F_{\theta_i}}{F_t} \right)^2 \right)^{(n-1)/2} F^{n-1} |F_t|\, dt,$$

for each j. Observe also that for all θ, if $t \in V(\theta)$, then $f'(t) = F_t(t,\theta)$ exists, so we have $\mathcal{H}^1(V(\theta)) = 1$ (see the first paragraph of this proof). Summing over j and using (36), we conclude that

$$\begin{aligned}\int_{RB\setminus\{u=1\}} |\nabla u|^{n-1}\, dx &= \int_{S^{n-1}} \int_{V(\theta)} \left(\frac{1}{(F_t)^2} + \sum_{i=1}^{n-1} \frac{k_i(\theta)}{F^2} \left(\frac{F_{\theta_i}}{F_t} \right)^2 \right)^{(n-1)/2} F^{n-1} |F_t|\, dt\, j(\theta)\, d\theta \\ &= \int_{S^{n-1}} \int_0^1 I_F(t,\theta)\, dt\, j(\theta)\, d\theta, \end{aligned} \tag{37}$$

where

$$I_F(t,\theta) = \frac{(F^2 + \sum_{i=1}^{n-1} k_i(\theta)(F_{\theta_i})^2)^{(n-1)/2}}{|F_t|^{n-2}} \tag{38}$$

for almost all $(t,\theta) \in (0,1) \times S^{n-1}$.

We now establish the corresponding formula for w^R. More care is needed in this case, since we do not know a priori that $|\nabla w^R|^{n-1}$ is integrable on $2RB \setminus (C \mathbin{\widetilde{+}} D)$. It follows from Lemma 4.5 and [9, p. 81] that w^R is differentiable almost everywhere in $\mathrm{int}(2RB) \setminus (C \mathbin{\widetilde{+}} D)$, and throughout the following discussion we shall work modulo the set of points of measure zero where w^R is not differentiable. It was established above that the partial derivatives of $\rho_{E^{R,t}}$ with respect to t and θ_i, $i = 1, \dots, n-1$, exist almost everywhere. Now taking $u = w^R$ and $F(t,\theta) = \rho_{E^{R,t}}(\theta)$, the change to polar coordinates via (31) and (32) gives, as before,

$$\int_{2RB\setminus(C\widetilde{+}D)} |\nabla u|^{n-1}\, dx = \int_{S^{n-1}} \int_{F(1,\theta)}^{2R} \left(u_\rho^2 + \sum_{i=1}^{n-1} \frac{k_i(\theta)}{\rho^2} u_{\theta_i}^2 \right)^{(n-1)/2} \rho^{n-1}\, d\rho\, j(\theta)\, d\theta. \tag{39}$$

Fix $\theta \in S^{n-1}$, and as above, let $g(\rho) = u(\rho\theta)$. Then g is strictly decreasing on $(F(1,\theta), 2R)$. Defining $f(t) = F(t,\theta) = \rho_{E^{R,t}}(\theta)$, we have that (33) holds and hence $f(t)$ for $t \in [0,1]$ is the inverse of g. By (18), $f'(t)$ exists for $t \in V(\theta) = V_C(\theta) \cap V_D(\theta)$. Moreover, on each of the countable family of disjoint open intervals whose union is $V(\theta)$, we see from (18) again that f is locally absolutely continuous, being the sum of locally absolutely continuous functions. Finally, we observe that by Lemma 4.5, for any $\varepsilon > 0$, $|\nabla w^R(\rho\theta)|$ is bounded for $\rho \in (F(1,\theta) + \varepsilon, 2R - \varepsilon)$, so the inner integral on the right-hand side of (39) is finite over this restricted interval. Now the change of variables via (34) and (35), which is again justified by [37, Corollary 6 and p. 519], yields

$$\int_{F(1,\theta)+\varepsilon}^{2R-\varepsilon} \left|\nabla w^R(\rho\theta)\right|^{n-1} \rho^{n-1}\, d\rho = \int_{W(\theta)} I_F(t,\theta)\, dt, \tag{40}$$

where for almost all (t,θ), I_F is given by (38) with $F = \rho_{E^{R,t}}$, and $W(\theta) = \{t \in V(\theta)\colon \rho_{E^{R,t}}(t,\theta) \in (F(1,\theta)+\varepsilon, 2R-\varepsilon)\}$.

Now for almost all (t,θ), we can apply Lemma 4.6 with $x_0 = |(\rho_{C^{R,t}})_t| > 0$, $y_0 = |(\rho_{D^{R,t}})_t| > 0$, $x_i = \sqrt{k_i(\theta)}(\rho_{C^{R,t}})_{\theta_i}(\theta)$, and $y_i = \sqrt{k_i(\theta)}(\rho_{D^{R,t}})_{\theta_i}(\theta)$, $i = 1,\dots,n$, and $x_n = \rho_{C^{R,t}}(\theta)$ and $y_n = \rho_{D^{R,t}}(\theta)$ in (28) to obtain

$$I_{\rho_{E^{R,t}}}(t,\theta) \leqslant I_{\rho_{C^{R,t}}}(t,\theta) + I_{\rho_{D^{R,t}}}(t,\theta). \tag{41}$$

Integrating with respect to t, we see that for each $\varepsilon > 0$ the integral in (40) is bounded above by

$$\int_0^1 I_{\rho_{C^{R,t}}}(t,\theta)\,dt + \int_0^1 I_{\rho_{D^{R,t}}}(t,\theta)\,dt,$$

which we know is finite for almost all θ. It follows that for almost all θ, (40) holds when $\varepsilon = 0$ and $W(\theta)$ is replaced by $V(\theta)$. Integrating the resulting equation with respect to θ and noting that $\mathcal{H}^1(V(\theta)) = 1$, we obtain (37) with $u = w^R$, $F = \rho_{E^{R,t}}$, and $RB \setminus \{u = 1\}$ replaced by $2RB \setminus (C \,\tilde{+}\, D)$. Therefore integration of (41) over $(0,1) \times S^{n-1}$ yields

$$\int_{2RB\setminus(C\tilde{+}D)} \left|\nabla w^R\right|^{n-1} dx \leqslant \int_{RB\setminus C} \left|\nabla u_C^R\right|^{n-1} dx + \int_{RB\setminus D} \left|\nabla u_D^R\right|^{n-1} dx. \tag{42}$$

Recalling that $u_C^R \in W^{1,n-1}(\mathrm{int}(RB) \setminus C)$ and $u_D^R \in W^{1,n-1}(\mathrm{int}(RB) \setminus D)$, we obtain from (42) that $w^R \in W^{1,n-1}(\mathrm{int}(2RB) \setminus (C \,\tilde{+}\, D))$. We also know that w_R is continuous, $w_R = 1$ on $\partial(C \,\tilde{+}\, D)$, $w_R = 0$ on ∂RB, and $C \,\tilde{+}\, D$ is Lipschitz, so $w^R \in W_0^{1,n-1}(\mathrm{int}(2RB))$. Since $w_R = 1$ in $C \,\tilde{+}\, D$, we obtain (29).

The argument for (30) is essentially the same as that used to obtain (29); it is only necessary to replace u_C^R, u_D^R, w^R, $C^{R,t}$, $D^{R,t}$, and $E^{R,t}$ by u_C, u_D, w, C^t, D^t, and E^t, respectively, and make the corresponding minor changes in the proof. □

Lemma 4.8. *Let $\varepsilon, R > 0$ and let C and D be bodies in $\mathbb{R}^n$, $n \geqslant 3$, contained in* $\mathrm{int}(RB)$ *and star-shaped with respect to εB. Let w be defined by* (19) *and* (20). *Then w is admissible for the $(n-1)$-capacity of $C \,\tilde{+}\, D$.*

Proof. In view of (30), we have $|\nabla w| \in L^{n-1}(\mathbb{R}^n)$, so it suffices to show that $w \in L^{n(n-1)}(\mathbb{R}^n)$. To this end, we first recall that $u_C^R, u_D^R \in W_0^{1,n-1}(RB)$. In the proof of Lemma 4.7, it was shown that $w^R \in W_0^{1,n-1}(2RB)$. Then by the Sobolev inequality (see [20, Theorem 7.10] for example), we have

$$\int_{2RB} \left|w^R\right|^{n(n-1)} dx \leqslant c_n \int_{2RB} \left|\nabla w^R\right|^{(n-1)} dx \tag{43}$$

where c_n depends only on n. By the C^1 convergence on compact subsets of u_C^R and u_D^R to u_C and u_D, respectively, the right-hand side of (29) converges to $\mathrm{Cap}_{n-1}(C) + \mathrm{Cap}_{n-1}(D)$ as $R \to \infty$. Combining this with (29) and (43), we obtain

$$\int_F \left|w^R\right|^{n(n-1)} dx \leqslant 2c_n\bigl(\mathrm{Cap}_{n-1}(C) + \mathrm{Cap}_{n-1}(D)\bigr),$$

for any compact set F in $\mathbb{R}^n$ and sufficiently large R. Letting $R \to \infty$ and using the uniform convergence of w^R to w on compact subsets established in Lemma 4.5, we obtain the same inequality for w, and this proves that $w \in L^{n(n-1)}(\mathbb{R}^n)$. □

Proof of Theorem 4.1. We begin by assuming that C and D are bodies in $\mathbb{R}^n$ star-shaped with respect to εB for some $\varepsilon > 0$. Under this assumption we shall obtain (12) and the equality condition. Then we shall describe how [3, Theorem 3.1] yields (12) for compact domains.

Since u_C and u_D are the $(n-1)$-capacitary functions of C and D (cf. (15)), we have

$$\mathrm{Cap}_{n-1}(C) = \int_{\mathbb{R}^n} |\nabla u_C|^{n-1}\, dx \quad \text{and} \quad \mathrm{Cap}_{n-1}(D) = \int_{\mathbb{R}^n} |\nabla u_D|^{n-1}\, dx. \tag{44}$$

By Lemma 4.8,

$$\mathrm{Cap}_{n-1}(C \,\widetilde{+}\, D) \leqslant \int_{\mathbb{R}^n} |\nabla w|^{n-1}\, dx. \tag{45}$$

Now (30), (44), and (45) yield (12).

Suppose that equality holds in (12). Then, by (44), (30), and (45), equality also holds in (30). Now (30) is obtained by integrating

$$I_{\rho_{E^t}}(t,\theta) \leqslant I_{\rho_{C^t}}(t,\theta) + I_{\rho_{D^t}}(t,\theta), \tag{46}$$

and this inequality holds for almost all (t,θ). Therefore equality must hold in (46) for almost all (t,θ). Since (46) was obtained in the proof of Lemma 4.7 by an application of Lemma 4.6, we see that the equality condition of Lemma 4.6 implies that for almost all (t,θ), there is an $\alpha(t,\theta) > 0$ such that $(\rho_{C^t})_{\theta_i}(\theta) = \alpha(t,\theta)(\rho_{D^t})_{\theta_i}(\theta)$, $i = 1, \ldots, n-1$, and $\rho_{C^t}(\theta) = \alpha(t,\theta)\rho_{D^t}(\theta)$. (Note that $x_n = \rho_{C^t}(\theta) > 0$ and $y_n = \rho_{D^t}(\theta) > 0$.) Therefore

$$\frac{(\rho_{C^t})_{\theta_i}(\theta)}{\rho_{C^t}(\theta)} = \frac{(\rho_{D^t})_{\theta_i}(\theta)}{\rho_{D^t}(\theta)},$$

for almost all (t,θ) and each $i = 1, \ldots, n-1$. Integrating with respect to θ_i and noting that ρ_{C^t} and ρ_{D^t} are Lipschitz functions of θ, we obtain $\rho_{C^t} = \beta(t)\rho_{D^t}$ for some $\beta(t) > 0$. Letting $t \to 1$ and using the fact that ρ_{C^t} and ρ_{D^t} are continuous in t, we see that ρ_{C^1} is a constant multiple of ρ_{D^1}. Recalling that $C^1 = C$ and $D^1 = D$ by Lemma 4.4, we conclude C is a dilatate of D.

We now describe how [3, Theorem 3.1] gives (12) when C and D are compact domains. Suppose initially that both C and D contain the origin in their interiors. In [3, Theorem 3.1] we take $\alpha = n-1$ (the parameter α corresponds to our p) and the function $g(r) = r^{n-1-\alpha} = 1$, noting that in this case their function $h(r) = 1$ also. In [3, Theorem 3.1] we also take $k = 2$, $a_1 = a_2 = 1/2$, and choose $\varepsilon > 0$ sufficiently small that we can set $\Omega_1 = \operatorname{int} C \setminus \varepsilon B$ and $\Omega_2 = \operatorname{int} D \setminus \varepsilon B$. Then [3, Theorem 3.1] states that

$$\mathrm{Cap}_{n-1}\left(\frac{1}{2}(\mathrm{int}\, C \setminus \varepsilon B) \mathbin{\widetilde{+}} \frac{1}{2}(\mathrm{int}\, D \setminus \varepsilon B)\right) \leqslant \frac{1}{2}\mathrm{Cap}_{n-1}(\mathrm{int}\, C \setminus \varepsilon B) + \frac{1}{2}\mathrm{Cap}_{n-1}(\mathrm{int}\, D \setminus \varepsilon B).$$

Since Cap_p is homogeneous of degree $n-p$ (see [9, Theorem 2(iv), p. 151]), we may remove the factors of $1/2$ in the previous inequality. We then allow $\varepsilon \to 0$, and use the definition (11) to obtain (12) when C and D contain the origin in their interiors. Now suppose that C and D are arbitrary compact domains. If C does not contain the origin in its interior, let $x \in C$ be a point nearest to o, and let $C_m = C \cup ([o,x] + (1/m)B)$ for $m \in \mathbb{N}$, where it is possible that $x = o$. Let D_m be defined analogously. Then C_m and D_m are compact domains containing the origin in their interiors, with $C \subset C_m$ and $D \subset D_m$. Since $\mathrm{Cap}_{n-1}([o,x]) = 0$ by [9, Theorem 3, p. 154], and in view of the subadditivity and monotonicity properties of Cap_{n-1} (see [9, Theorem 2(vii) and (ix), p. 151]), we have $\mathrm{Cap}_{n-1}(C_m) \to \mathrm{Cap}_{n-1}(C)$ and $\mathrm{Cap}_{n-1}(D_m) \to \mathrm{Cap}_{n-1}(D)$ as $m \to \infty$. The fact that (12) holds for C_m and D_m allows us to conclude that it also holds for C and D. □

5. Radial sums and surface area

Lemma 5.1. *If C and D are Lipschitz star bodies in $\mathbb{R}^2$, then*

$$S(C \mathbin{\widetilde{+}} D) \leqslant S(C) + S(D),$$

with equality if and only if C is a dilatate of D.

Proof. For each Lipschitz star body M in $\mathbb{R}^2$, we have

$$S(M) = \int_0^{2\pi} \sqrt{\rho_M(\theta)^2 + \rho'_M(\theta)^2}\, d\theta.$$

By definition,

$$\rho_{C \widetilde{+} D} = \rho_C + \rho_D,$$

so that

$$\rho'_{C \widetilde{+} D}(\theta) = \rho'_C(\theta) + \rho'_D(\theta),$$

for almost all θ. For $a, b, c, d \in \mathbb{R}$, we have the inequality

$$\sqrt{(a+b)^2 + (c+d)^2} \leqslant \sqrt{a^2 + c^2} + \sqrt{b^2 + d^2},$$

with equality if and only if $bc = ad$. Using these facts, we obtain

$$S(C \mathbin{\widetilde{+}} D) = \int_0^{2\pi} \sqrt{\big(\rho_C(\theta) + \rho_D(\theta)\big)^2 + \big(\rho'_C(\theta) + \rho'_D(\theta)\big)^2}\, d\theta$$

$$\leqslant \int_0^{2\pi} \sqrt{\rho_C(\theta)^2 + \rho_C'(\theta)^2}\, d\theta + \int_0^{2\pi} \sqrt{\rho_D(\theta)^2 + \rho_D'(\theta)^2}\, d\theta$$
$$= S(C) + S(D).$$

Equality holds if and only if $\rho_D(\theta)\rho_C'(\theta) = \rho_C(\theta)\rho_D'(\theta)$ for almost all θ. Rearranging the previous equation, integrating, and using the fact that $\log \rho_C$ and $\log \rho_D$ are absolutely continuous, we obtain $\log \rho_C = \log \rho_D + c$ for some constant c. Then ρ_C is a positive multiple of ρ_D, so C is a dilatate of D. □

Theorem 5.2. *If C and D are star bodies in $\mathbb{R}^2$, then*

$$S(C \,\widetilde{+}\, D) \leqslant S(C) + S(D).$$

Proof. Let M be a star body in $\mathbb{R}^2$. Then ∂M is a closed curve, and

$$S(M) = \mathcal{H}^1(\partial M) = \sup\bigl\{S(P):\ P \text{ is a polygonal star body whose vertices lie in } \partial M\bigr\}.$$

See, for example, [26, Theorem 3.2.4]. Let P be a polygonal star body whose vertices lie in $\partial(C \,\widetilde{+}\, D)$, and let $V = \{v_1, \dots, v_m\}$ be the set of vertices of P ordered in increasing angle with the positive x-axis. For $1 \leqslant i \leqslant m$, let $v_i(C) = [o, v_i] \cap \partial C$ and $v_i(D) = [o, v_i] \cap \partial D$. Let Q (or R) be the polygonal star body whose boundary is the union of the line segments $[v_i(C), v_{i+1}(C)]$, $1 \leqslant i \leqslant m-1$, and $[v_m(C), v_1(C)]$ (or the line segments $[v_i(D), v_{i+1}(D)]$, $1 \leqslant i \leqslant m-1$, and $[v_m(D), v_1(D)]$, respectively). Then Q and R are polygonal star bodies whose vertices lie in ∂C and ∂D, respectively.

The star body $Q \,\widetilde{+}\, R$ contains V in its boundary, which is a union of arcs A_i with endpoints v_i and v_{i+1}, $1 \leqslant i \leqslant m-1$, and A_{m+1} with endpoints v_m and v_1. The length of each arc is at least the length of the line segment in the boundary of P with the same endpoints. Consequently, by Lemma 5.1 with C and D replaced by Q and R, respectively, we have

$$S(P) \leqslant S(Q \,\widetilde{+}\, R) \leqslant S(Q) + S(R) \leqslant S(C) + S(D).$$

Since P was arbitrary, the theorem is proved. □

In view of Theorem 5.2, it is natural to ask whether

$$S(C \,\widetilde{+}\, D)^{1/(n-1)} \leqslant S(C)^{1/(n-1)} + S(D)^{1/(n-1)} \tag{47}$$

for Lipschitz star bodies C and D in $\mathbb{R}^n$, $n \geqslant 3$. Note that the exponent in (47) could not be replaced by any larger number. To see this, let $C = aB$ and $D = bB$ for $a, b > 0$. If (47) holds with $1/(n-1)$ replaced by $p > 0$, we have, by the homogeneity of surface area,

$$(a+b)^{p(n-1)} \leqslant a^{p(n-1)} + b^{p(n-1)}.$$

If $q > 0$ and $a, b > 0$, the *qth sums* $(a^q + b^q)^{1/q}$ decrease with q (see [21, (2.10.5), p. 29]). Therefore $p \leqslant 1/(n-1)$.

However, (47) is false in general for all $n \geqslant 3$. Indeed, we shall prove that the much weaker inequality

$$S\left(\frac{1}{2}C \mathbin{\tilde{+}} \frac{1}{2}D\right) \leqslant \frac{1}{2}S(C) + \frac{1}{2}S(D)$$

is also false in general for all $n \geqslant 3$. (To see that this is indeed weaker, replace C and D in (47) by $(1/2)C$ and $(1/2)D$, respectively, and use the homogeneity of $S^{1/(n-1)}$ and the fact (see [21, Section 2.9]) that *pth means* $((a^p + b^p)/2)^{1/p}$ increase with p.) To prove this, we need the following lemma. We omit the routine proof.

Lemma 5.3. *Let C be a star body of revolution about the x_n-axis in $\mathbb{R}^n$. If φ is the vertical spherical polar angle, then*

$$S(C) = (n-1)\kappa_{n-1} \int_0^{\pi} \rho_C(\varphi)^{n-2} \sqrt{\rho_C(\varphi)^2 + \rho_C'(\varphi)^2} \sin^{n-2}(\varphi)\, d\varphi.$$

Theorem 5.4. *Let $n \geqslant 3$. There is a star body C of revolution about the x_n-axis in $\mathbb{R}^n$ such that*

$$S\left(\frac{1}{2}C \mathbin{\tilde{+}} \frac{1}{2}B\right) > \frac{1}{2}S(C) + \frac{1}{2}S(B).$$

Proof. Let $M > 0$ and let $m \in \mathbb{N}$ be such that $0 < M\pi/(2m) = \varepsilon < 1/2$. Let $f(\varphi)$, $0 \leqslant \varphi \leqslant \pi$, be a sawtooth function satisfying $\varepsilon \leqslant f(\varphi) \leqslant 2\varepsilon$, $0 \leqslant \varphi \leqslant \pi$, and $|f'(\varphi)| = M$ for all $\varphi \neq j\pi/(2m)$, $j = 0, \dots, 2m$. Let C be the star body of revolution about the x_n-axis in $\mathbb{R}^n$ whose radial function is given by $\rho_C(\varphi) = f(\varphi)$. (A meridian section of C can be visualized as a cookie-cutter-shaped planar star body.) Then, by Lemma 5.3, we obtain

$$S(C) \leqslant (n-1)\kappa_{n-1} I_n (2\varepsilon)^{n-2} \sqrt{4\varepsilon^2 + M^2} < (n-1)\kappa_{n-1} I_n (2\varepsilon)^{n-2} \sqrt{1 + M^2},$$

where

$$I_n = \int_0^{\pi} \sin^{n-2}(\varphi)\, d\varphi,$$

and

$$\begin{aligned} S\left(\frac{1}{2}C \mathbin{\tilde{+}} \frac{1}{2}B\right) &\geqslant (n-1)\kappa_{n-1} I_n \left(\frac{1+\varepsilon}{2}\right)^{n-2} \sqrt{\left(\frac{1+\varepsilon}{2}\right)^2 + \left(\frac{M}{2}\right)^2} \\ &> (n-1)\kappa_{n-1} I_n \frac{\sqrt{1+M^2}}{2^{n-1}}. \end{aligned}$$

Since $S(B) = (n-1)\kappa_{n-1} I_n$ by Lemma 5.3, it suffices to choose ε and M so that

$$\sqrt{1+M^2}\left(\frac{1}{2^{n-2}}-(2\varepsilon)^{n-2}\right)>1,$$

and the quantity in parentheses is positive. This can be done, for example, by taking $m=8M$ and $M\in\mathbb{N}$ sufficiently large. □

Note that the construction in the proof of the previous theorem also shows that (47) is false in general when the exponent $1/(n-1)$ is replaced by any $p>0$.

Another possible generalization of Lemma 5.1 (and of Theorem 5.2) is as follows. If C is a Borel star set in $\mathbb{R}^n$, let S_C be given in spherical polar coordinates by

$$S_C=\left\{\left(\rho_C(u),u\right):\ u\in S^{n-1}\right\},$$

and let $S(C)=\mathcal{H}^{n-1}(S_C)$. If C is a Lipschitz star body, then $S(C)$ is the surface area of C. Then we can ask whether (47) holds when $n=2$ and C and D are Borel star sets.

6. Radial sums and 1-capacity

The following theorem is a generalization of the case $n=2$ of [29, Section 8], where it is shown that $\mathrm{Cap}_1(K)=S(K)$ for a convex body K in $\mathbb{R}^n$.

Lemma 6.1. *If C is a connected compact set in $\mathbb{R}^2$, then*

$$\mathrm{Cap}_1(C)=S(\operatorname{conv}C).$$

Proof. Let E be a body with C^∞ boundary containing C. If F is the component of E containing C, then $S(F)\leqslant S(E)$. Since F is a subset of $\mathbb{R}^2$, we clearly have $S(\operatorname{conv}F)\leqslant S(F)$. By the continuity of surface area in the class of compact convex sets (a consequence of the continuity of mixed volumes; see [15, p. 399] or [40, p. 295]), for each $\varepsilon>0$ there is a convex body K containing $\operatorname{conv}F$ that has a C^∞ boundary and satisfies

$$S(K)\leqslant S(\operatorname{conv}F)+\varepsilon\leqslant S(E)+\varepsilon.$$

It follows from Proposition 2.2 that

$$\mathrm{Cap}_1(C)=\inf\left\{S(K)\right\},$$

where the infimum is taken over all convex bodies K with C^∞ boundary containing C. By [36, p. 160], we can choose a sequence $\{K_i\}$, $i\in\mathbb{N}$, of such bodies with $K_i\to\operatorname{conv}C$ as $i\to\infty$ in the Hausdorff metric. By the continuity of surface area in the class of compact convex sets again, $S(K_i)\to S(\operatorname{conv}C)$ as $i\to\infty$, and the result follows. □

Note that the word "connected" cannot be omitted in Lemma 6.1. For example, the union of the vertices of a triangle has 1-capacity zero by Proposition 2.1. When $n>2$, Lemma 6.1 is false even for Lipschitz star bodies of revolution. To see this, let $a>0$ and let E in $\mathbb{R}^n$ be the union of the line segment $[-ae_n,ae_n]$, where e_n is a unit vector parallel to the x_n axis, and the unit ball D of dimension $n-1$ in $\{x\in\mathbb{R}^n:\ x_n=0\}$. The 1-capacity of a line segment is zero, $\mathrm{Cap}_1(D) = S(D) = 2\kappa_{n-1}$, and 1-capacity is monotonic and subadditive, so $\mathrm{Cap}_1(E) = 2\kappa_{n-1}$. Let $K = \mathrm{conv}\, E$ and let M_i, $i \in \mathbb{N}$, be a decreasing sequence of Lipschitz star bodies of revolution about the x_n-axis in $\mathbb{R}^n$ containing E and converging to E in the Hausdorff metric as $i \to \infty$, so that $\mathrm{conv}\, M_i \to K$ as $i \to \infty$. Then $S(\mathrm{conv}(M_i)) \to S(K)$ as $i \to \infty$, and by [9, Theorem 2(v) and (ix), p. 151], $\mathrm{Cap}_1(M_i) \to \mathrm{Cap}_1(E) = 2\kappa_{n-1}$ as $i \to \infty$. But $S(K) > 2\kappa_{n-1}$. Thus it is not possible that $\mathrm{Cap}_1(M_i) = S(\mathrm{conv}(M_i))$ if i is sufficiently large.

Lemma 6.2. *If K and L are convex bodies containing the origin in $\mathbb{R}^2$, then*

$$S\big(\mathrm{conv}(K \,\widetilde{+}\, L)\big) \leqslant S(K) + S(L), \tag{48}$$

with equality if and only if K is a dilatate of L.

Proof. Suppose first that the compact star set $K \,\widetilde{+}\, L$ is a body (it need not be, as we observed in Section 3). Then (48) follows directly from Lemma 5.1 and the easily proved fact that if M is a body in $\mathbb{R}^2$, then $S(\mathrm{conv}\, M) \leqslant S(M)$.

Suppose, then, that $K \,\widetilde{+}\, L$ is not a body. This can only occur when both ∂K and ∂L contain line segments contained in the same line through the origin (and hence also $o \in \partial K$ and $o \in \partial L$), and the interiors of K and L have empty intersection. Then $K' = K + \varepsilon B$ and $L' = L + \varepsilon B$, where $\varepsilon > 0$, must be such that $K' \,\widetilde{+}\, L'$ is a body, and so $S(\mathrm{conv}(K' \,\widetilde{+}\, L')) \leqslant S(K') + S(L')$. Letting $\varepsilon \to 0$ and using the continuity of surface area in the class of compact convex sets, we obtain (48) also in this case.

Suppose that equality holds in (48). It is easy to see that $K \,\widetilde{+}\, L$ must be a body. If K and L contain the origin in their interiors, the equality condition follows from that of Lemma 5.1. Otherwise, we have either $o \in \partial K$ or $o \in \partial L$, or both. Moreover, since equality holds in the inequality $S(\mathrm{conv}\, M) \leqslant S(M)$ if and only if the body M is convex, we conclude that $K \,\widetilde{+}\, L$ must also be a convex body. Then the supports of the radial functions of K and L, i.e. the closures of the sets of θ for which $\rho_K(\theta) > 0$ or $\rho_L(\theta) > 0$, must coincide in a common interval, $[\alpha, \beta]$, say. Exactly as in the proof of Lemma 5.1, we conclude that ρ_K is a positive multiple of ρ_L on $[\alpha, \beta]$ and it follows that K must be a dilatate of L. $\square$

Lemma 6.3. *If K is a convex body in $\mathbb{R}^2$, then*

$$S(K^*) \leqslant S(K),$$

with equality if and only if K contains the origin.

Proof. If K contains the origin, then $K^* = K$ and there is nothing to prove. Suppose, then, that $o \notin K$. Let $L = \mathrm{conv}\{o, K\}$ and let M be the closure of $L \setminus K$. Then L and M are compact star sets and we have

$$\rho_{K^*}(\theta) = \rho_L(\theta) - \rho_M(\theta),$$

for θ in the common support, $[\alpha, \beta]$, say, of ρ_L and ρ_M. Similarly to the proof of Lemma 5.1, but this time using the inequality

$$\sqrt{(a-b)^2 + (c-d)^2} \leqslant \sqrt{a^2 + c^2} + \sqrt{b^2 + d^2},$$

for $a \geqslant b$ and $c \geqslant d$, we obtain

$$S(K^*) \leqslant \int_\alpha^\beta \sqrt{\rho_L(\theta)^2 + \rho_L'(\theta)^2}\, d\theta + \int_\alpha^\beta \sqrt{\rho_M(\theta)^2 + \rho_M'(\theta)^2}\, d\theta.$$

Since the right-hand side of the previous inequality is just $S(K)$, the required inequality is proved, and in this case it is easy to see that strict inequality must hold. □

Theorem 6.4. *If C and D are compact domains in $\mathbb{R}^2$, then*

$$\mathrm{Cap}_1(C \mathbin{\tilde{+}} D) \leqslant \mathrm{Cap}_1(C) + \mathrm{Cap}_1(D). \tag{49}$$

If C and D are star sets, equality holds if and only if $\operatorname{conv} C$ *is a dilatate of* $\operatorname{conv} D$.

Proof. Let C and D be compact domains in $\mathbb{R}^2$. By the definition (11) of radial sum, we have

$$C \mathbin{\tilde{+}} D = C^* \mathbin{\tilde{+}} D^* \subset (\operatorname{conv} C)^* \mathbin{\tilde{+}} (\operatorname{conv} D)^* = \operatorname{conv} C \mathbin{\tilde{+}} \operatorname{conv} D, \tag{50}$$

and hence

$$\operatorname{conv}(C^* \mathbin{\tilde{+}} D^*) \subset \operatorname{conv}\big((\operatorname{conv} C)^* \mathbin{\tilde{+}} (\operatorname{conv} D)^*\big). \tag{51}$$

By the monotonicity of surface area in the class of convex bodies (a consequence of the monotonicity of mixed volumes; see [15, p. 399]), we have

$$S\big(\operatorname{conv}(C^* \mathbin{\tilde{+}} D^*)\big) \leqslant S\big(\operatorname{conv}\big((\operatorname{conv} C)^* \mathbin{\tilde{+}} (\operatorname{conv} D)^*\big)\big). \tag{52}$$

We now use definition (11), Lemma 6.1, (52), and Lemmas 6.2 and 6.3 to obtain

$$\begin{aligned}
\mathrm{Cap}_1(C \mathbin{\tilde{+}} D) = \mathrm{Cap}_1(C^* \mathbin{\tilde{+}} D^*) &= S\big(\operatorname{conv}(C^* \mathbin{\tilde{+}} D^*)\big) \\
&\leqslant S\big(\operatorname{conv}\big((\operatorname{conv} C)^* \mathbin{\tilde{+}} (\operatorname{conv} D)^*\big)\big) \\
&\leqslant S\big((\operatorname{conv} C)^*\big) + S\big((\operatorname{conv} D)^*\big) \\
&\leqslant S(\operatorname{conv} C) + S(\operatorname{conv} D) \\
&= \mathrm{Cap}_1(C) + \mathrm{Cap}_1(D).
\end{aligned}$$

Suppose that equality holds in (49) for *arbitrary* compact domains C and D. Then equality must also hold in Lemma 6.2, with K and L replaced by $(\operatorname{conv} C)^*$ and $(\operatorname{conv} D)^*$, respectively (the latter being convex bodies, as was noted in Section 3). Therefore $(\operatorname{conv} C)^*$ is a dilatate of $(\operatorname{conv} D)^*$. Equality must also hold in Lemma 6.3, with K replaced by either $\operatorname{conv} C$ or $\operatorname{conv} D$, so both $\operatorname{conv} C$ and $\operatorname{conv} D$ contain the origin. This also yields $\operatorname{conv} C = (\operatorname{conv} C)^*$ and $\operatorname{conv} D = (\operatorname{conv} D)^*$ and hence $\operatorname{conv} C$ is a dilatate of $\operatorname{conv} D$.

Suppose that C and D are star sets and $\operatorname{conv} C$ is a dilatate of $\operatorname{conv} D$. Let z be an extreme point of the convex set $\operatorname{conv} C \mathbin{\tilde{+}} \operatorname{conv} D$. Then $z = x + y$, where x is an extreme point of $\operatorname{conv} C$ and y is an extreme point of $\operatorname{conv} D$. Therefore $x \in C$ and $y \in D$, so $z \in C \mathbin{\tilde{+}} D$. This shows that

$$\operatorname{conv} C \,\widetilde{+}\, \operatorname{conv} D \subset \operatorname{conv}(C \,\widetilde{+}\, D).$$

Since $\operatorname{conv} C$ is a dilatate of $\operatorname{conv} D$, $\operatorname{conv} C \,\widetilde{+}\, \operatorname{conv} D$ is convex and then (50) implies the opposite inclusion. Therefore

$$\operatorname{conv}(C \,\widetilde{+}\, D) = \operatorname{conv} C \,\widetilde{+}\, \operatorname{conv} D.$$

By the equality condition of Lemma 6.2 with K and L replaced by $\operatorname{conv} C$ and $\operatorname{conv} D$, respectively, this implies that equality holds in (49). □

The proof of Theorem 6.4 explicitly states that if equality holds in (49), then $\operatorname{conv} C$ and $\operatorname{conv} D$ contain the origin and $\operatorname{conv} C$ is a dilatate of $\operatorname{conv} D$. The proof also shows that for equality in (49) it is necessary and sufficient that in addition equality must hold in (52) and hence, in view of (51), that

$$\operatorname{conv}(C^* \,\widetilde{+}\, D^*) = \operatorname{conv}\big((\operatorname{conv} C)^* \,\widetilde{+}\, (\operatorname{conv} D)^*\big). \tag{53}$$

If C and D are not star sets, it is less transparent how to turn this into a neat geometric condition. Suppose that D is a dilatate of C. If C is a "horseshoe" containing the origin in the interior of one its ends, then (53) does not hold, while it does so if $C = [-1, 1]^2 \setminus (\operatorname{int}(\varepsilon B) + (1 - 2\varepsilon, 0))$ for small $\varepsilon > 0$ (an origin-symmetric square with a small hole near the middle of one of its edges).

As we mentioned in Section 1, it remains open whether for compact domains C and D in $\mathbb{R}^n$ and $1 \leqslant p < n$ with $p \neq n - 1$, it is true that

$$\operatorname{Cap}_1(C \,\widetilde{+}\, D)^{1/(n-p)} \leqslant \operatorname{Cap}_1(C)^{1/(n-p)} + \operatorname{Cap}_1(D)^{1/(n-p)}. \tag{54}$$

By the same reasoning as was given after (47), (54) cannot hold for any exponent larger than $1/(n-p)$.

In particular, we do not know if (54) is true when $p = 1$ and $n \geqslant 3$. In view of the fact that $\operatorname{Cap}_1(K) = S(K)$ for a convex body K, however, we remark that the inequality

$$S\big(\operatorname{conv}(K \,\widetilde{+}\, L)\big)^{1/(n-1)} \leqslant S(K)^{1/(n-1)} + S(L)^{1/(n-1)}$$

does not hold in general when K and L are convex bodies in $\mathbb{R}^n$. An easy calculation shows that is false, for example, when K is the standard octahedron in $\mathbb{R}^3$ with vertices at $(\pm 1, 0, 0)$, $(0, \pm 1, 0)$, and $(0, 0, \pm 1)$ and L is an octahedron with vertices at $(\pm a, 0, 0)$, $(0, \pm a, 0)$, and $(0, 0, \pm 1)$ for $0 < a < 1$.

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